Applications of Integration

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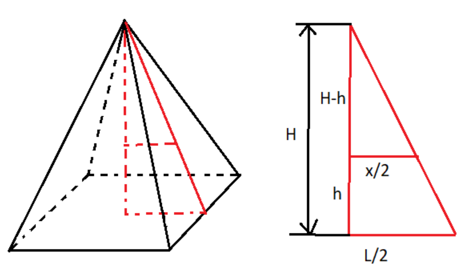
1 Volume by Sections

We have seen how to compute the volume of a ball by considering horizontal sections. That is, we make a partition of the ball into many small pieces by horizontal cutting, and then take the Riemann sum of the volume of the cylinders that are used for approximating the actual volume. The Riemann sum on one hand is the actual volume of the ball, and on the other hand is a definite integral. This method can be generalized to the following theorem:

Theorem 1. *E is an 3-dimensional object of height H. The area of the section of E at height h is $S(h)$, then the volume of E is*

$$\int_0^H S(h)\,dh$$

Example 2. *Compute the volume of a pyramid whose height is H and base is a square with each edge having length L.*



At height h, by similar triangles, we can find the length of each edge of the section to be $x = \frac{H-h}{H}L$. So by the formula, the volume is

$$\begin{aligned}\int_0^H (\frac{H-h}{H}L)^2\,dh &= \frac{L^2}{H^2}\int_0^H (H-h)^2\,dh \\ &= \frac{L^2}{H^2}\int_0^H (h-H)^2\,d(h-H) \\ &= \frac{L^2}{H^2}\frac{(h-H)^3}{3}\Big|_0^H \\ &= \frac{1}{3}HL^2\end{aligned}$$

A special case is when the volume is generated from rotation of a curve:

Theorem 3. *Given the graph of $y = f(x)$ on the xy-plane with domain $[a, b]$, if we rotate the graph about x-axis in the 3-dimensional space, the graph sweeps over a surface. The volume bounded inside this surface is given by*

$$\pi\int_a^b f(x)^2\,dx$$

Example 4. *Compute the volume of the solid obtained from rotating $y = \frac{1}{x}$ along x-axis between $[1, 2]$.*

$$\begin{aligned}\pi\int_1^2 (\frac{1}{x})^2\,dx &= \pi\int_1^2 \frac{1}{x^2}\,dx \\ &= -\pi\frac{1}{x}\Big|_1^2 \\ &= \frac{\pi}{2}\end{aligned}$$

2 Consumer and Producer Surplus

We have often used demand and supply functions in economics to study the equilibrium price of some goods. The equilibrium price is the one at which the demand equals the supply, which will be the actual price that appear in market.

But there are some rich consumers who are able and willing to buy even if the actual price were higher than the equilibrium price. In this case, it is the market that saves money for them. We are going to study how much money is saved for them, and we call it the consumer surplus. We will define it in a mathematical way.

Let (P^*, Q^*) be the equilibrium point.

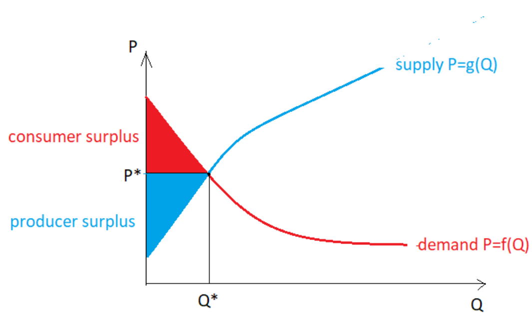
Assume the demand function is $P = f(Q)$. If consumers already bought the amount of Q units, then if the demand increases by a small ΔQ, the total extra amount of money consumers pay is about $f(Q)\Delta Q$, since a small change in Q won't influence the change of price by a lot. So the standard Riemann sum argument implies if the price starts from $P_0 = f(0)$, then during the process of obtaining equilibrium price, consumers would have spent in total

$$\int_0^{Q^*} f(Q)\, dQ$$

dollars, which reflects the total amount of money that consumers are willing to pay before reaching equilibrium price. But the actual amount of money they pay based on the equilibrium price is $f(Q^*)Q^*$, so the market help people to save

$$\int_0^{Q^*} f(Q) - f(Q^*)\, dQ = \int_0^{Q^*} f(Q) - P^*\, dQ$$

in total.



Similarly, if we consider the supply, there are sellers who are happy to sell at price lower than the equilibrium. Let $P = g(Q)$ be the supply function.

A similar argument as above shows that the market help sellers to earn an extra

$$\int_0^{Q^*} g(Q^*) - g(Q)\, dQ$$

This amount is called producer surplus.

3 Distribution Function

When some data is given on a large population, we can develop a method that can show us the percentage of population within any given interval in a fast way, making use of integration.

For example, consider the income of people in a population. For each income r, define $F(r)$ to be the proportion of the population whose income is less than r. Note that F is a discrete function, since income is only defined in the accuracy of 0.01 dollars. But in the case when the population is big, we can often find a differentiable function that is a good approximation of F, so we may assume F is differentiable. Let $f(r) = F'(r)$, the derivative of F. By the Fundamental Theorem of Calculus, we know

$$F(b) - F(a) = \int_a^b f(r)\, dr$$

By the definition of F, $F(b) - F(a)$ stands for the proportion of the population whose height is between a and b, which can be computed from definite integrals of f. F is called the **cumulative distribution function**, and f is called the **density function**.

The next question is to consider the mean income. Assume there are in total N people. Given an income interval $[a, b]$, we divide $[a, b]$ into n pieces of length $\Delta r = \frac{b-a}{n}$. When n is big, Δr is very small, so $f(r) = F'(r) \approx \frac{F(r_i)-F(r_{i-1})}{\Delta r}$, which implies $F(r_i) - F(r_{i-1}) \approx f(r)\Delta r$, we thus get the total amount of people in the range $[r_{i-1}, r_i]$ is approximated by $Nf(r_i)\Delta r$. The total income made by people whose income is in $[r_{i-1}, r_i]$ can be approximated by $Nrf(r_i)\Delta r$. So Taking the limit of the sum $\sum_{i=1}^{n} Nrf(r_i)\Delta r$, we see the total income is

$$\int_a^b Nrf(r)\, dr = N \int_a^b rf(r)\, dr$$

The total number of people on this income range is given by

$$N \int_a^b f(r)\, dr$$

So we see the mean income on this interval is

$$\frac{N \int_a^b r f(r)\, dr}{N \int_a^b f(r)\, dr} = \frac{\int_a^b r f(r)\, dr}{\int_a^b f(r)\, dr}$$

Remark 5. By the definition of F, we see that $\lim_{x\to+\infty} F(x) = 1$, and $F(0) = 0$. Since $F(b) - F(0) = \int_0^b f(x)\, dx$, we conclude

$$\lim_{x\to+\infty} \int_0^b f(x)\, dx = 1$$

In calculus, the above limit is also expressed as

$$\int_0^{+\infty} f(x)\, dx = 1$$

which is called the improper integral.

In Economics, a Pareto function is one that described the income distribution. The proportion of people that earn at most r dollars is given by $f(r) = \frac{B}{r^\beta}$ for some constants B, β.

Example 6. *Consider a population of n individuals in which the income density function for those with income between a and b is*

$$f(r) = \frac{B}{r^{2.5}}$$

where B is a constant. Find the mean income of the group, and approximate it when b is very big.

First, we see the total number of people whose income is between a and b is

$$n \int_a^b f(r)\, dr = n \int_a^b \frac{B}{r^{2.5}}\, dr = \frac{2}{3} nB(a^{-1.5} - b^{-1.5})$$

Next, we need to figure out the total income of these people. This is given by

$$n\int_a^b rf(r)\,dr = n\int_a^b \frac{B}{r^{1.5}}\,dr = 2nB(a^{0.5} - b^{0.5})$$

So the average income is given by

$$\frac{2nB(a^{0.5} - b^{0.5})}{\frac{2}{3}nB(a^{-1.5} - b^{-1.5})} = 3\frac{a^{-0.5} - b^{-0.5}}{a^{-1.5} - b^{-1.5}}$$

When b is very big, $b^{0.5} \to 0$ and $b^{1.5} \to 0$, so the mean is approximated by

$$3\frac{a^{-0.5}}{a^{-1.5}} = 3a$$